GUNDELFINGER'S CONIC SECTIONS.

Vorlesungen aus der Analytischen Geometrie der Kegelschnitte. Von SIGMUND GUNDELFINGER. Herausgegeben Von FRIEDRICH DINGELDEY. Leipzig, Teubner, 1895. pp. 434.

AMONG recent analytic works on Conic Sections there are at least three for which one is very thankful, — the late Professor Casey's, Miss Scott's, and the one whose title appears above. The plan of this last is to systematically develop the theory by means of homogeneous coordinates, while bringing out the fact that the elementary (x, y) system is merely a case to which we can descend when so minded. This latter may seem a minor point; pedagogically it is not so, and it is certainly not well explained in many books.

The development, then, of the theory is really analytic, though one feels that the analysis is under the control of a masterly geometric insight.

The work divides into two sections and an appendix. The first section begins with the explanation of point-coordinates in all their generality. In spite of the fact that two of the three books above mentioned adopt this plan, it does seem a good method to explain these matters — at all events at first — with a special system of coordinates or "unit-point," and with the "areal" ("barycentric") system for preference, we get the full advantage of a homogeneous system, awkward factors do not appear, and if in any projective question we need an arbitrary unit-point, we have merely to show how to project the fundamental triangle so that the unit-point projects into the centroid. Evidently there is a pedagogic gain, but it is not evident that there is any loss.

We have then the fundamental expression for the distance of a point from a line; and the inference that when the point and line unite, the expression

$$u_1x_1 + u_2x_2 + u_3x_3$$

is zero, whence we have the equation of a line. It is necessary to mention such a detail so long as elementary text-books reverse the order and prove *first* that the equation of a line (p, α) is

$$x\cos\alpha + y\sin\alpha - p = 0,$$

and *second* that the left represents the distance from the line (p, α) to the point (x, y).

The coordinates of a line are next defined as the coefficients u_1, u_2, u_3, and their connection with the distances from the fundamental points explained.

In the next article the angle between two lines is determined and brought into the form

$$\cos^{-1}\frac{\omega(u, v)}{\sqrt{\omega(u, u)}\sqrt{\omega(v, v)}},$$

and the ambiguity arising from the square roots on the one hand and the choice of angles on the other is resolved. The formula for the distance of two points is given much later.

For the curve of the second order $f(x, x)=0$ (§ 4) it is shown first that it cuts out two points from an arbitrary range $x+y$; next by making the points coincide, the equation of the pair of tangents from y is determined; next comes the double ratios of the two intersections on the one hand, and x, y on the other, and the all-important case $f(x, y)=0$, when the two pairs are harmonic, is discussed.

From this, by letting y lie on its polar $f(x, y)=0$, the equation of the tangent is deduced. It is to be observed that the tangent is regarded primarily as a line which meets the curve in coincident points, *not* as a line which unites with its pole.

Soon the discriminant and the line-equation of $f(x, x)=0$ are introduced, and the condition that two lines u_n, v_n meet on the curve is found in the form

$$\begin{pmatrix} uv \\ uv \end{pmatrix}=0$$

where the notation expresses the determinant obtained by bordering the discriminant with u_1, u_2, u_3 and v_1, v_2, v_3. When u is given, this is the equation of the two points where u meets the conic, and the resolution of $\begin{pmatrix} uv \\ uv \end{pmatrix}$ into factors is actually effected. This is Gundelfinger's method, and another method (due to Aronhold) is explained whereby also the intersections are expressed in terms of the coordinates of the intersecting line.

In the next article the curve of the second class is similarly handled, without unnecessary repetition of algebra.

Then follows (§ 6) the classification of the conic sections, and this is gone into with fulness.

In § 7 the circle and its points at infinity make their appearance, the former through a line-equation, as the envelope of a line whose distance from a fixed point is constant. We notice that Miss Scott pursues the same plan. The points at infinity then appear of their own accord, as points into whose equations

the equation $\omega(u, u) = 0$ breaks up. Then comes the point-equation of a circle of given centre and radius, and thereby of course the distance between two points. Returning to the points at infinity, it is shown that the expression $\omega(u, u)$ is a "definite" form; that is, it cannot change sign for real values of u_1, u_2, u_3. It would perhaps have been better to have established this point by connecting the expression with the squared area of the fundamental triangle. And lastly the angle made by one line with another is connected with a double ratio made by the points at infinity and the circular points.

The following article is devoted to algebra, and contains developments based on Kronecker and Weierstrass of well-known theorems. First a general theorem on "definite" quadratic forms is proved, namely that *if* $\phi(u_1, u_2 \cdots u_n)$ *is such a form, which vanishes for the values* $p_1, p_2 \cdots p_n$ *of the variables, of which values at least one is not zero, then the expressions* $\phi'(p_1), \phi'(p_2) \cdots \phi'(p_n)$ *also all vanish.* This theorem is applied to prove that what Salmon (*Higher Algebra*, lesson vi.) calls the "equation of secular inequalities," namely,

$$\begin{vmatrix} a_{11} - \lambda, & a_{12}, & \cdots a_{1n} \\ a_{21}, & a_{22} - \lambda, & \cdots a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1}, & a_{n2}, & \cdots a_{nn} - \lambda \end{vmatrix} = 0,$$

where $a_{ik} = a_{ki}$, has only real roots; and further to prove that if the determinant has a factor $(\lambda - \lambda_1)^t$, then the minor of any term has at least the factor $(\lambda - \lambda_1)^{t-1}$ (when $\lambda_1 \lesseqgtr 0$), and at least the factor λ^{t-2} (when $\lambda_1 = 0$).

In § 9 is an introductory account of invariants, covariants, contravariants, and "mixed concomitants," and it concludes with the reduction of $f(x, x) = 0$ to two terms only, namely, to

$$x_1 x_3 - c x_2^2 = 0,$$

or, as the equation is later written,

$$x_2^2 + 2 x_1 x_3 = 0,$$

a form which has the advantage of being its own reciprocal. In § 10 the general equation is transformed so as to be referred to the principal axes, with the data necessary for determining the equations and lengths of these axes, etc. Herein the previous algebraic section finds its use, but one cannot avoid regretting the fact that 24 pages are required for a detailed treatment of the transformation.

The next article discusses the forms of the curves from the equations referred to their axes, finds the foci by combining

the line-equation with the imaginary circular points, and proves the focus and directrix property, and the fact that the sum or difference of focal radii for the ellipse or hyperbola is constant. The proofs are given always with a view to the general handling of the homogeneous equation; the section is by no means an idle incursion into the field of the elementary text-book.

In § 12 we return to projective problems, and have the generation of a conic by projective pencils or ranges, the remarkable theorem of Pascal and the reciprocal theorem of Brianchon. The developments of Kirkman and others are not gone into.

The second section of the work deals with pencils and nets of conics, and the reciprocal ideas of ranges and webs.

First the three line-pairs of a pencil are found, and the cubic, written

$$\lambda^3 B - 3\lambda^2\Theta + 3\lambda H - A = 0,$$

which leads to them is carefully discussed. Then it is shown that the double points of these line-pairs form a triad harmonic with all conics of the pencil.

Then, to mention only leading propositions, which will suffice to show the order of ideas, the two conics of a pencil which touch a given line as determined. Thence follows Desargue's theorem that a line is cut in involution by the conics of a pencil. Then it is proved that a line, which cuts from two given conics point-pairs whose double ratios are assigned, envelops a curve of the fourth class, and thence in particular Staudt's conic, the envelope when the point-pairs are harmonic, makes its appearance.

The reciprocal statements as to a range are then made; and special reference is made to the range of confocal conics; *i.e.* the range which includes the circular points. Desargue's theorem, or rather its reciprocal, shows that the two conics through any point are confocal, and Staudt's conic, taken for the circular points and any conic, appears as the director circle of that conic.

In the next place suitable triangles of reference are assigned for the special forms which a pencil assumes owing to coincidences among the ground-points.

Then comes the significance of the vanishing of the intermediate invariants of two conics, the Θ and H of the cubic before mentioned; and the case where one or both of the conics degenerates is not overlooked. Conics whose Θ vanishes are termed conjugate, instead of apolar as with Reye.

The confocal range has now an article to itself, with full treatment of the possible special cases. The important property of the foci, that the product of their distances from any tangent is constant, begins a new article, wherein are discussed the circles having double contact with a given conic.

The next article returns to the pencil of conics, and proves the theorem that the poles of a given line, as to all the conics, lie on a conic N, and discusses especially the case when the given line is the line ∞. The reciprocal proposition that the polars of a given point y, as to a range of conics, envelop a conic N, is applied to a confocal range; the envelope is now called Steiner's parabola. In particular is shown the bearing of this parabola on the problem of the four normals from y to a conic of the range; the feet of the normals are the points of contact with the selected conic of common lines of it and of the Steiner's parabola which arises from the point y.

When the parabola is reciprocated with respect to the conic, it is shown analytically that we have a hyperbola through the centre of the conic and the points at ∞ on its axes. That this rectangular hyperbola — which Casey calls the Apollonian hyperbola — also solves the problem of drawing normals from a point to a conic, is shown by means of an elegant expression for a curve which intersects a curve of degree n at the feet of the normals from a point y. Leaving special considerations, it is then shown that when the conic N is written

$$N_1u_1 + N_2u_2 + N_3u_3 = 0,$$

where u is the given line, the Jacobian of N_1, N_2, N_3 as to x_1, x_2, x_3 is nothing else than the sides of the diagonal triangle of the four ground-points of the pencil. And the article ends with a too brief account of a remarkable combinant conic $\psi=0$, which Casey (*Conics*, second edition, p. 486) calls the fourteen-line conic. As this conic is not perhaps yet well known, it may be worth while to briefly indicate it. Consider the four points a, b, c, d, whose coordinates are $\lambda_1, \pm\lambda_2, \pm\lambda_3$, and the four lines $\alpha, \beta, \gamma, \delta$, whose coordinates are $1/\lambda_1, \pm 1/\lambda_2, \pm 1/\lambda_3$. The fundamental triangle is the common harmonic triangle of the four points and the four lines, and the two systems of four elements have to one another a well-known elementary and perfectly reciprocal projective relation. The conic whose point-equation is

$$x_1^2/\lambda_1^2 + x_2^2/\lambda_2^2 + x_3^2/\lambda_3^2 = 0,$$

and whose line-equation is

$$\lambda_1^2\xi_1^2 + \lambda_2^2\xi_2^2 + \lambda_3^2\xi_3^2 = 0$$

is the conic in question. On each of the four lines are three points, where it is met by the other three lines; the Hessian pair of these points are on the conic. On each side of the fundamental triangle are two point-pairs, such as $\beta\gamma$, $\alpha\delta$, $bd \mid ac$, and $cd \mid ab$ (where $bd \mid ac$ means the intersection of bd and ac). The Jacobian of these two point-pairs, which of course

are harmonic pairs, is two points of the conic. Thus the fourteen points are enumerated, and the fourteen lines follow by interchange of point and line. The equations show that when we take four real points or lines the conic is imaginary.

The conic is considered further, in connection with the two "equianharmonic" conics through the four points, on p. 370. The three are written

$$x_1^2 + e x_2^2 + e^2 x_3^2 = 0,$$
$$e^3 = 1.$$

Perhaps Wolstenholme's equivalent form,

$$x_1^2 + 2x_2x_3 = 0,\ \ x_2^2 + 2x_3x_1 = 0,\ \ x_3^2 + 2x_1x_2 = 0,$$*

is more convenient for the treatment of the system. The conic, or rather the combinant which, put zero, gives the conic, affords great service in the discussion (p. 371) of the reality of the intersections of two conics whose point-equations are assigned. The complete solution of this problem is given in a form more simple than that of Kemmer.

Returning to complete our sketch of the order of the book, the radius of curvature, which was mentioned in connection with Steiner's parabola, is found for any algebraic curve whose line-equation is assigned, and the result stated also for a curve of assigned point-equation. The evolute of a conic is also discussed.

The next three articles (§§ 22–25) discuss the Hessian and Cayleyan of a net of conics, and prove (so far as proof is needed) the reciprocal theorems as to the Hessian and Cayleyan of a web. It must be observed that what is here called the Hessian is called by Salmon the Jacobian of the three base-conics (SALMON, *Conics*, p. 360). Especially to be noticed is the method of § 25.

The following will indicate the main result of the article. Consider the net

$$y_1(x_1^2 + 2mx_2x_3) + y_2(x_2^2 + 2mx_3x_1) + y_3(x_3^2 + 2mx_1x_2) = 0,$$

to which form any net may be reduced unless it contain a double line† and the web

$$v_1(u_1^2 + 2\mu u_2u_3) + v_2(u_2^2 + 2\mu u_3u_1) + v_3(u_3^2 + 2\mu u_1u_2) = 0.$$

The condition of conjugacy ($\Theta = 0$) is

$$(y_1v_1 + y_2v_2 + y_3v_3)(1 + 2m\mu) = 0;$$

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* BULLETIN, Series 2, vol. 1, p. 121.

† SALMON, *Conics*, pp. 365, 367. In the equation at top of p. 367, the coefficient m should be inserted.

if, then, $1 + 2m\mu = 0$, each member of the net is conjugate with each member of the web.

On the other hand, considering the cubics

$$x_1^3 + x_2^3 + x_3^3 + 6mx_1x_2x_3 = 0,$$
$$u_1^3 + u_2^3 + u_3^3 + 6\mu u_1u_2u_3 = 0,$$

whose polar conics and pole conics respectively constitute the net and web, the Hessian of the former is (SALMON, *Curves*, p. 190)

$$-m^2(x_1^3 + x_2^3 + x_3^3) + (1 + 2m^3)x_1x_2x_3 = 0,$$

and the Cayleyan of the latter (*ib.* p. 191) is

$$\mu(x_1^3 + x_2^3 + x_3^3) + (1 - 4\mu^3)u_1u_2u_3 = 0.$$

These are the same if

$$-1/m^2 - 2/m = 1/\mu - 4/\mu^2,$$

and in particular if

$$1 + 2m\mu = 0.$$

Thus if we call the net and web "conjugate" (or apolar) *the Hessian of a net is the Cayleyan of the conjugate web; and reciprocally the Hessian of the web is the Cayleyan of the net.* But the more careful treatment of our book appears necessary in view of the special cases.

The Hessian and Cayleyan are thus the same thing in different aspects; and this idea is fruitfully applied to the Hessian and Cayleyan of a net or of a web.

In the final article (§ 26), the relation of conjugacy is further developed, and applied (for instance) to prove that between the squares of six tangents of a conic there is a linear relation.

Then follows an appendix of 180 pages, consisting for the most part of solved examples, which sufficiently demonstrate the applicability and completeness of the theory previously developed.

A number of problems are taken from Steiner, who, it will be recalled, stated various results without proof. These results of Steiner's are not all, at the present date, very obvious; and the solutions, which are partly due to Dr. Gundelfinger, partly to Dr. Dingeldey, are both direct and elegant. It is, I think, fair to say that until the appearance of the books named at the beginning, students were well able to handle conics with homogeneous point-coordinates, and to reciprocate the theorems arrived at; but the use of point and line coordinates combined was not fairly presented. For this reason (not to mention others) these solutions will repay the attention of students.

A point of historical interest in the Geometry of the Triangle should be mentioned. It appears that Steiner came across the

"cosine circle" of a triangle (p. 316), that is, he noticed as a special case of one of his theorems that there is only one point p (the point now called the symmedian) through which if lines be drawn so that the intercepts made on them by the pairs of sides of a triangle are bisected at p, the ends of the intercepts lie on a circle, whose centre is of course p.

Finally we have the handling of two integrals, the second of which

$$\int \frac{\Sigma \pm (c_1 x_2 dx_3)}{[\frac{1}{2} c_1 g'(x_1) + \frac{1}{2} c_2 g'(x_2) + \frac{1}{2} c_3 g'(x_3)]\sqrt{f(x, x)}}$$

(where $f(x, x)$ and $g(x, x)$ are ternary quadratic forms, x_1, x_2, x_3 are the coordinates of a point on $g(x, x) = 0$, c_1, c_2, c_3, the coordinates of a fixed point on the same conic), is transformed into

$$-\int \frac{d\lambda}{\sqrt{g(\lambda)}}$$

where $g(\lambda)$ is the cubic which determines the line-pairs of the pencil defined by f and g. This covers, for example, the problem of transforming the elliptic integral $\int (x dx)/\sqrt{a_x^4}$ to Weierstrass's normal form.

FRANK MORLEY.

ON DIVERGENT SERIES.

BY PROFESSOR A. S. CHESSIN.

THAT every semi-convergent series can by a proper arrangement of its terms be made divergent is a well-known fact. It will be shown in this note that, conversely, every divergent series which does not tend towards infinity (series oscillating between finite limits) can by a proper arrangement of its terms be made convergent.

Only series with real terms will be considered since the investigation of series with complex terms, at least with regard to the substance of this note, can be reduced to that of series with only real terms.

THEOREM I. — *An infinity of numbers being given within a limited interval, we know that there will be at least one infinite accumulation of numbers of the given totality within the given interval. Let, in general, $N_1, N_2, \cdots, N_g$ be the numbers about which these infinite accumulations take place. It is always possible to form g distinct convergent series having for their respective sums the numbers $N_1, N_2, N_3, \cdots, N_g$.*

In fact, in the neighborhood of a number N_i there is an infinity of numbers of the given totality either on both sides of N_i or only on one side of it. In the last case, if $\alpha_{\kappa,i}$ be one of these numbers, we can always find another one among them $\alpha_{\kappa+1,i}$ such that $\alpha_{\kappa+1,i} > \alpha_{\kappa,i}$ or $\alpha_{\kappa+1,i} < \alpha_{\kappa,i}$ according as these numbers are on one side of N_i or on the other. We thus obtain a *regular* sequence of increasing or decreasing numbers such that $\lim(\alpha_i) = N_i$. In the other case we can always find two numbers $\alpha_{1,i}$ and $\beta_{1,i}$ among the given ones in the neighborhood of the number N_i such that there will be an infinity of numbers of the given totality within the interval $(\alpha_{1,i}\, \beta_{1,i})$. In this interval we can again find two numbers $\alpha_{2,i}\, \beta_{2,i}$ such that $\alpha_{2,i} > \alpha_{1,i}$ and $\beta_2 < \beta_{1,i}$ and that the interval $(\alpha_{2,i}\, \beta_{2,i})$ shall contain an infinity of numbers of the given totality. Continuing this process, we arrive at two *regular* sequences of numbers

$$\alpha_{1,i}\, \alpha_{2,i}\, \alpha_{3,i} \cdots$$
$$\beta_{1,i}\, \beta_{2,i}\, \beta_{3,i} \cdots$$

the first having only increasing, the second only decreasing terms. Both define the same number N_i

i.e. $$\lim(\alpha_i) = \lim(\beta_i) = N_i.$$

Let us now put

$$\alpha_{1,i} = a_{1,i}$$
$$\alpha_{2,i} - \alpha_{1,i} = a_{2,i}$$
$$\alpha_{3,i} - \alpha_{2,i} = a_{3,i}$$
$$\cdots \quad \cdots$$

then $$\alpha_{n,i} = a_{1,i} + a_{2,i} + \cdots + a_{n,i}$$

and therefore

$$N_i = a_{1,i} + a_{2,i} + \cdots + a_{n,i} + \cdots$$

for all the values $1, 2, 3, \cdots, g$ of i; q. e. d.

Suppose now that we have a divergent series which does not tend towards infinity

$$u_1 + u_2 + \cdots + u_n + \cdots,$$

and let us form the sequence of numbers

$$\gamma_1, \gamma_2, \cdots, \gamma_n, \cdots$$

where $\gamma_m = u_1 + u_2 + \cdots + u_m$. This sequence contains an infinity of numbers within a limited interval. Let then

$$N_1, N_2, \cdots, N_g$$

be the numbers about which infinite accumulations of the numbers (γ) take place. According to Theorem I. we can form g convergent series having for their respective sums the numbers $N_1, N_2, \cdots, N_g$.

THEOREM II.— *The g convergent series derived from the sequence (γ) can be obtained directly from the given divergent series merely by associating its terms in a proper way* (without deranging their places in the series).

In fact, we have seen in Theorem I. that we can pick out among the numbers (γ) in the neighborhood of the number N_i an infinity of numbers (γ) forming a regular sequence

$$\gamma_{\mu_{1,i}}, \gamma_{\mu_{2,i}}, \cdots, \gamma_{\mu_{m,i}}, \cdots,$$

and that $\lim (\gamma_{\mu_i}) = N_i$, for all the values $1, 2, \cdots, g$ of i.

In this sequence there is at least one index $\mu_{k,i} > \mu_{1,i}$; for otherwise the number of terms following the term $\gamma_{\mu_{1,i}}$ would be at the most equal to $\mu_{1,i}$. In like manner we prove that there must be at least one index $\mu_{e,i} > \mu_{k,i}$ and so on. We thus obtain the regular sequence

$$\gamma_{\mu_{1,i}}, \gamma_{\mu_{k,i}}, \gamma_{\mu_{e,i}}, \cdots$$

in which $\mu_{1,i} < \mu_{k,i} < \mu_{e,i} < \cdots$; and $\lim (\gamma_{\mu_i}) = N_i$.

But $$\gamma_{\mu_{e,i}} = u_1 + u_2 + \cdots + u_{\mu_{e,i}}.$$

If therefore we associate the terms of the given series in the following manner:

$$(u_1 + u_2 + \cdots + u_{\mu_{1,i}}) + (u_{\mu_{1,i}+1} + \cdots + u_{\mu_{k,i}}) + (u_{\mu_{k,i}+1} + \cdots + u_{\mu_{e,i}}) + \cdots,$$

this series will be convergent and have for its sum the number N_i; q. e. d.

Remark. — It is obvious that g is always greater than unity, the numbers N_i being all finite, otherwise the given series (u) would be convergent.

Examples:

(1) In the famous series

$$1 - 1 + 1 - 1 + 1 - \cdots$$

$$g = 2;\ N_1 = 0 = (1 - 1) + (1 - 1) + \cdots$$

$$N_2 = 1 = 1 + (-1 + 1) + (-1 + 1) + \cdots$$

(2) In the series

$$\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{2}{3} - \tfrac{3}{4} + \tfrac{3}{4} - \tfrac{4}{5} + \cdots$$

$$g = 2;\ N_1 = -\tfrac{1}{2} = (\tfrac{1}{2} - \tfrac{2}{3}) + (\tfrac{2}{3} - \tfrac{3}{4}) + (\tfrac{3}{4} - \tfrac{4}{5}) + \cdots$$

$$N_2 = \tfrac{1}{2} = \tfrac{1}{2} + (-\tfrac{2}{3} + \tfrac{2}{3}) + (-\tfrac{3}{4} + \tfrac{3}{4}) + \cdots$$

A divergent series which remains divergent whatever be the arrangement of its terms will be called *unconditionally divergent.*

THEOREM III. — *Every unconditionally divergent series tends towards infinity.*

In fact, in such a series $g = 1$, but this is impossible in a divergent series unless $N_1 = \pm \infty$.

It is easy to see that *conversely, every series which tends towards infinity is unconditionally divergent.*

A divergent series which becomes convergent after a proper arrangement of its terms will be called *conditionally divergent.*

THEOREM IV. — *Every divergent series which does not tend towards infinity is conditionally divergent, and conversely, a conditionally divergent series cannot tend towards infinity.*

THEOREM V. — *Every semi-convergent series can by a proper arrangement of its terms be made conditionally divergent, and conversely, every conditionally divergent series can by a proper arrangement of its terms be made semi-convergent.*

Remark. — Riemann has proved that by a proper arrangement (commutative, not associative) of the terms of a semi-convergent series this series can be made to converge to any arbitrarily assigned number. It follows from the above that *by a proper arrangement* (both commutative and associative) *of the terms of a conditionally divergent series this series can be made to converge to any arbitrarily assigned number.*

THE JOHNS HOPKINS UNIVERSITY,
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A SIMPLE PROOF OF A FUNDAMENTAL THEOREM OF SUBSTITUTION GROUPS, AND SEVERAL APPLICATIONS OF THE THEOREM.

BY DR. G. A. MILLER.

THEOREM. — *The average number of elements in all the substitutions of a group is $n - \alpha$, n being the degree of the group, and α the number of its transitive constituents.**

We shall first prove the theorem for $\alpha = 1$, *i.e.* for the transitive groups.

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* FROBENIUS, *Crelle*, vol. 101, p. 287.

Let g represent the order of such a group, and $a_1, a_2, \ldots, a_n$ its elements. Since the group contains only g/n substitutions* that do not involve a_α for each particular value of α in the series $\alpha = 1, 2, \cdots, n$, it must contain $g - g/n = \frac{n-1}{n} g$ substitutions that contain any one of the n given a_α's. Hence all the substitutions of the group contain $\frac{n-1}{n} gn = (n-1)g$ elements. As the number of substitutions is g, the average number of elements in all of them must be $n-1$.

To see that the theorem is also true when $\alpha > 1$, *i.e.* when the group is intransitive, it is only necessary to observe that every intransitive group may be resolved into transitive constituent groups, and that the substitutions of these transitive constituents are equally distributed among the substitutions of the intransitive group.†

Applications.

For every substitution of a transitive group which contains less than $n-1$ elements there must be a sufficient number of substitutions of the nth degree to make the average number of elements in all these substitutions $n-1$. In particular, for identity there must be $n-1$ substitutions of the nth degree, since $\frac{0 + (n-1)n}{n} = n - 1$. Hence the formula‡

$$n_n = n - 1 + (n-3)n_2 + (n-4)n_3 + \cdots + n_{n-2},$$

n_α representing the number of substitutions in the group, which contain exactly α letters.

This formula is equivalent to the following:§

$$n_n = n\left(\tfrac{1}{2} n'_{n-2} + \cdots + \frac{n-\alpha-1}{n-\alpha} n'_\alpha + \cdots + \frac{n-1}{n} n'_0\right).$$

Here n'_α bears the same relation to the subgroup which contains all the substitutions that replace a given letter by itself as n_α bears to the entire group.

Let H be the subgroup which is generated by all the substitutions of the nth degree in any transitive group G. If H were intransitive, the average number of elements in all its substitutions would, according to the given theorem, be less

* NETTO'S Theory of Substitutions (COLE'S translation), § 61.
† BOLZA, *American Journal*, vol. 11, p. 200.
‡ CAUCHY, *Comptes Rendus*, vol. 21 (2), p. 897.
§ JORDAN, *Comptes Rendus*, vol. 74, p. 977.

than $n-1$, while the average number of elements in all the substitutions of G is $n-1$. This is impossible, as the substitutions of G which are not also in H cannot contain more than $n-1$ elements. *Hence H is transitive.* Since the average number of elements in the substitutions of G is the same as that in the substitutions of H, *the substitutions of G which are not found in H must all be of the $(n-1)$th degree.*

The theorems of §§ 75, 76 of Netto's work * are proved by the preceding paragraph. It may be well to add in regard to the given § 75 that G may be transitive while the corresponding subgroup of G is intransitive. The following group † is an instance:

1	1	$AB.CE$	$af.bj.ci.dh.eg$
$ABCDE$	$abcde.fghij$	$AC.DE$	$ag.bf.cj.di.eh$
$ACEBD$	$acebd.fhjgi$	$AD.BC$	$ah.bg.cf.dj.ei$
$ADBEC$	$adbec.fiqjh$	$AE.BD$	$ai.bh.cg.df.ej$
$AEDCB$	$aedcb.fjihg$	$BE.CD$	$aj.bi.ch.dg.ef$

Netto's statement: "If G is transitive in the A's, H is transitive in the x's," together with the rest of the section seems to me to imply that such a case is impossible.

LEIPZIG, *August*, 1895.

ON AN UNDEMONSTRATED THEOREM OF THE DISQUISITIONES ARITHMETICÆ.

BY DR. JAMES PIERPONT.

THE last section of the Disquisitiones Arithmeticæ contains the algebraic solution of the equations upon which the prime roots of unity depend.

$$x^p - 1 = 0, \ddagger \tag{1}$$

where p is a prime. As this equation contains the factor $(x-1)$, we may consider instead the equation

$$x^{p-1} + x^{p-2} + \cdots + x + 1 = 0. \tag{2}$$

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* COLE'S translation, pp. 86, 87.

† COLE, *Quarterly Journal*, vol. 27, p. 41.

‡ The algebraic solution of these equations so simple in form presented difficulties which the mathematicians of the last century were not able to surmount. When $p = 11$ one arrives at an equation of 5th degree. Vandermonde gave the solution of this equation at the close of his paper Mémoire sur la Résolution des Equations, Hist. Acad. de Paris, 1771, but it appears to have been little known.

Gauss shows how the roots of this equation may be rationally expressed in the roots of the *suite*

$$Z_1 = 0, \quad Z_2 = 0, \cdots, \tag{3}$$

whose coefficients are respectively rational in the roots of the preceding equations, the coefficients of the first being integers. The degrees of the equations (3) are precisely the prime factors of $p-1$; hence it follows that if $p-1$ contains no factors other than 2, the solution of (1) can be effected by the extraction of square roots of known quantities. Now the roots of (1) expressed in circular functions are

$$x_\kappa = \cos\frac{2\pi\kappa}{p} + i\sin\frac{2\pi\kappa}{p}, \quad \kappa = 0, 1, \cdots, p-1,$$

to which correspond in the plane representing the complex variable precisely the vertices of a regular polygon of p sides. Further, since the roots of a quadratic equation whose coefficients can be constructed geometrically, *i.e.* by rule and compass, can also be constructed geometrically, it follows that when $p-1$ is of the form 2^μ, the roots of the *suite* (3) can be constructed, and hence those of (1). Thus we are able to construct all polygons of a prime number of sides p, if $p-1$ is a power of 2. Such primes are

$$3, \quad 5, \quad 17, \quad 257, \quad 65537, \cdots.$$

Of this result Gauss remarks* that it is certainly remarkable that although the geometric construction of regular polygons of 3 and 5 sides and those immediately derivable from them, viz.

$$2^\mu, \quad 2^\mu \cdot 3, \quad 2^\mu \cdot 5, \quad 2^\mu \cdot 15,$$

was already known in Euclid's time, still in an interval of 2000 years not only no new polygons had been discovered, but geometers were unanimous in declaring no others could be constructed.

A result of this startling character could easily tempt Gauss's contemporaries to seek other constructible polygons. Against such an attempt, however, Gauss expressly warns in the following words:

"As often as $p-1$ contains other prime factors besides 2, we arrive at higher equations, namely, to one or more cubic equations, if 3 enters once or oftener as a factor of $p-1$, to equations of 5th degree if $p-1$ is divisible by 5, etc. And

* Gauss, Disq. Arith., Art. 365.

we can prove with all rigor that these higher equations cannot be avoided or made to depend upon equations of lower degree; and although the limits of this work do not permit us to give the demonstration here, we still thought it necessary to signal this fact in order that one should not seek to construct other polygons than those given by our theory, as, for example, polygons of 7, 11, 13, 19 sides, and so employ his time in vain."

Having laid down the theory for polygons of a prime number of sides, Gauss now turns his attention to polygons of any number of sides, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\nu^{\alpha_\nu}$ where $p_1, p_2, \cdots$, are the prime factors of n. These he disposes of in a very summary fashion by declaring, without any attempt of proof, that they can be constructed then and only then when

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_\nu}\right)$$

contains no other factor than 2.*

That this is a sufficient condition follows at once from an easy extension of Gauss's method as developed when n is a prime. It is, however, vastly more important to know that *only* these polygons can be geometrically constructed as thereby the theory of regular polygons, as far as their construction by rule and compass is concerned, is made complete.† That is, in a given case we can decide whether the polygon is constructable, and in case that it is, Gauss's theory gives us the necessary directions to construct it.

We propose now in the *first part* of this paper to show that the condition which Gauss gave as necessary is in fact such. The demonstration, resting as it does on only the most *elementary algebraic* notions, is thought will be of interest as filling up a *lacuna* which all readers of the Disquisitiones must have felt. In the *second part* we shall establish a general algebraical theorem from which follows at once the correctness of Gauss's statement ‡ made in regard to the number and degree of the equations entering into the solution of (1).

We make, then, a further application to establish a criterion regarding the construction of regular polygons by *rational* conic sections.

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* *Ibid.* Art. 366.

† This is the character of all Gauss's works. Compare his letter to Schumacher, Nov. 21, 1825, which contains the following passage: "Der Wunsch den ich immer bei meinen Arbeiten gehabt habe, ihnen eine solche Vollendung zu geben *ut nihil amplius desiderari possit* . . ."

‡ Put in their precise form. As we prove Gauss's criterion, it is not necessary to investigate these equations; it is, however, interesting to show that Gauss's assertions in this respect are correct.

I.

To show that the condition given by Gauss is necessary, we observe in the first place that if a quantity x can be constructed by rule and compass, it is rational in the roots of a *suite* of quadratic equations.

$$X_1 = 0,\ X_2 = 0, \cdots, \tag{4}$$

whose coefficients are respectively rational in the roots of the preceding equations, while those of $X_1 = 0$ are integers. This is simply shown by remarking that to each right line or circle which enters into the construction of x corresponds an equation respectively of the type

$$\frac{x}{a} + \frac{y}{b} = 1, \tag{5}$$

$$(x-a)^2 + (y-b)^2 = c^2,$$

where a, b, c are quantities which can be geometrically constructed. The solution of a succession of such equations leads to a *suite* of the type (4). Having established this analytical condition for the constructibility of a quantity, we proceed to show that a polygon of $m = p^a$ sides (p being a prime) cannot be constructed unless

$$\phi(m) = p^{a-1}(p-1)$$

is a power of 2. To do this we need only to show that the roots of

$$x^m - 1 = 0 \tag{6}$$

are not rational in a *suite* of the type (4). Instead, however, of considering (6) we consider the equivalent equation

$$F(x) = x^{(p-1)p^{a-1}} + x^{(p-2)p^{a-1}} + \cdots + x^{p^{a-1}} + 1 = 0, \tag{7}$$

whose roots are the $\phi(m)$ primitive mth roots of unity. Kronecker has shown * that $F(x)$ is irreducible, *i.e.* it is impossible to break $F(x)$ into two rational integral polynomials $\chi(x)$, $\psi(x)$ with rational coefficients.

For suppose $$F(x) = \chi(x)\psi(x),$$

then the coefficients of χ, ψ are not only rational but integers. As $F(1) = p$, one of the polynomials $\chi(1)$, for example, $= \pm 1$.

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* KRONECKER, Werke, vol. 1, p. 101. Kronecker proves here the irreducibility of $F(x)$ when m is a prime p; remarks, however, that the method is applicable to the above case.

Form now the product

$$P(x) = \chi(x)\,\chi(x^2)\cdots\chi(x^{m-1}).$$

As $P(x)$ is divisible by $F(x)$, we have

$$\frac{P(x)}{F(x)} = q(x),$$

whose coefficients are evidently integers. Hence $q(1)$ is an integer, which is impossible, since

$$q(1) = \frac{P(1)}{F(1)} = \frac{\pm 1}{p}.$$

Suppose now the roots of F are rational in a *suite* as (4); proceed to enlarge the domain $R(1)$* in which $F(x)$ is irreducible by successively adjoining the roots of the equations (4), which we denote respectively by

$$y_1, y_1'; \; y_2, y_2'; \cdots.$$

At some moment $F(x)$ breaks up into rational factors, say on adjoining y_κ, we can then write

$$F(x) = F_1(x, y_\kappa)\,F_2(x, y_\kappa)\cdots F_s(x, y_\kappa),$$

where $F_1, F_2 \cdots$ are rational polynomials having their coefficients rational in y_κ and preceding irrationalities. Let F_t be of lowest degree λ; then $m \gtreqless s\lambda$; further, the coefficients of

$$\Phi(x) = F_t(x, y_\kappa)\,F_t(x, y_\kappa'),$$

being symmetric in the roots of $X_\kappa = 0$, are rational in $R(y_1 y_2 \cdots y_{\kappa-1})$ in which domain $F(x)$ is irreducible. As $\Phi = 0$ is satisfied by at least one root of $F = 0$, Φ is divisible by F, and hence the degree of Φ, that is 2λ, $\lesseqgtr m \lesseqgtr s\lambda$, whence

$$s \gtreqless 2, \text{ or as } s > 1, \text{ we have } s = 2.$$

That is, when $F(x)$ becomes reducible, it breaks up into two factors of equal degree. As the same reasoning is applicable to $F_t(x, y_\kappa)$, etc., we conclude that it is impossible to break

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*To avoid circumlocution the author uses the terminology introduced by GALOIS; for those unfamiliar with it we add the following explanation: Let $abc\cdots$ be certain quantities; all quantities which can be derived from them by addition, subtraction, multiplication, and division, form a *domain* of rationality $R(a, b, c, \cdots)$. If α be not in R, we may enlarge R by *adjoining* it to $a, b, c, \cdots$, thus forming a new domain $R(a, b, c, \cdots \alpha)$. Thus $R(1)$ denotes all rational numbers. *It is to be remarked*, however, that while using Galois' terms, we make no use of his theory in Part I.; this would have destroyed the simplicity of the proof.

$F(x)$ into linear factors if $\phi(m)$ contains other prime factors than 2, which requires that

$$a = 1,\ p - 1 = 2^{\mu}. \tag{8}$$

From this we conclude immediately that a polygon of

$$n = p_1^{a_1} p_2^{a_2} \cdots p_\nu^{a_\nu}$$

sides cannot be geometrically constructed unless p_κ, $\kappa = 1, 2 \cdots \nu$, satisfy the condition (8); that is,

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_\nu}\right)$$

must be a power of 2.

This shows at once the impossibility of constructing by rule and compass polygons of 7, 9, 11, 13, 14 ··· sides.

As the ancient Greek geometers frequently allowed the use of the conic sections in a geometric construction, the question naturally arises, what additional polygons can be under these new conditions constructed? This will be answered below.

II.

Incidently we proved in the preceding section that an irreducible equation of degree n cannot be solved by a *suite* (4) of quadratic equations if n contains prime factors besides 2. The equations (4) are simple Abelian equations, and the theorem just stated is a particular case of the following:

THEOREM I. — Let the roots of an irreducible algebraically solvable equation $F(x) = 0$ of degree $n = p_1^{a_1} p_2^{a_2} \cdots p_\nu^{a_\nu}$, ($p_1 p_2 \cdots$ being primes) be rational in the roots of a *suite* of simple equations

$$f_1 = 0,\ f_2 = 0 \cdots \tag{9}$$

having the properties

1° The coefficients of $f_1 = 0$ are rational, while those of the following are respectively rational in the roots of the preceding equations.

2° The roots of no equation are rational in those of the preceding equations.

Then the *suite* (9) contains a_κ Abelian equations of degree p_κ ($\kappa = 1, 2 \cdots \nu$); should $F(x) = 0$ be itself Abelian, the *suite* (9) need contain no other equations.*

For proceed to enlarge R by the successive adjunction of the roots of (9); at a certain moment a reduction of the group

* Cf. an analogous theorem by HÖLDER, *Math. Annalen*, vol. 34, p. 26.

of $F=0$ must take place. Let it be on adjoining the roots of $f_\kappa=0$ of degree m_κ; then $f_\kappa=0$ is an Abelian equation of degree m_κ, and m_κ is a prime. If at the same time $F(x)$ becomes reducible, it breaks up into precisely m factors of equal degree. Since $F(x)$ ultimately breaks up into linear factors, it follows that (9) must contain at least a_κ Abelian equations of degree p_κ. If $F(x)=0$ is itself an irreducible Abelian equation, it need contain no more.

This theorem contains Gauss's statement in regard to the number and degree of the equations upon which the solution of (1) depends as a particular case. As another application of Theorem I., we conclude easily the following:

THEOREM II. — A polygon of n sides can then, and only then, be constructed by a series of *rational* conics* when $\phi(n)$ contains no prime factor other than 2 or 3.

For suppose $\phi(n)$ contain only such factors, then the solution of (1) depends upon a *suite*

$$Y_1=0,\quad Y_2=0,\cdots$$

of equations of the 2d or 3d degree. As the coefficients of each successive equation of this *suite* are constructible, the roots themselves are,† and hence the roots of (1). This shows that the above condition is sufficient; that it is necessary, follows from the same reasoning as employed for the circle in Part I.

We have deduced Theorem II. from Theorem I. We remark that it can be proved very simply by other considerations.

The following is a short table of constructible regular polygons of sides $\gtrless 100$.

The first row indicates the polygon constructible by rule and compass, known to the Greeks; the second row indicates the polygons of this class discovered by Gauss; finally, the last row gives the *additional* polygons which can be constructed when *rational conics* can be employed.

GREEKS:	2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 32, 40, 48, 60, 64, 80, 96.
GAUSS:	17, 34, 51, 68, 85.
CONICS:	7, 9, 13, 14, 18, 19, 21, 26, 27, 28, 35, 36, 37, 38, 39, 42, 45, 52, 54, 56, 57, 63, 65, 70, 72, 73, 74, 76, 78, 81, 84, 90, 91, 95, 97.

NEW HAVEN, *November*, 1895.

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* That is, conics whose coefficients are rational in the current domain of rationality.

† Cf. DESCARTES' Geometria, edit. by v. Schooten.

NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York, Saturday afternoon, November 30, at three o'clock, the President, Dr. HILL, in the chair. There were fifteen members present. On the recommendation of the Council, the following persons, nominated at the preceding meeting, were elected to membership: Professor ALEXANDRE S. CHESSIN, Johns Hopkins University, Baltimore, Md.; Dr. JONATHAN BRACE CHITTENDEN, Columbia College, New York; Mr. ARTHUR BOWES FRIZELL, University of City of New York, New York; Mr. JOHN ETHAN HILL, Columbia College, New York; Professor ELLEN HAYES, Wellesley College, Wellesley, Mass.; Professor THOMAS EMERY MCKINNEY, Marietta College, Marietta, O.; Professor EDGAR JEROME TOWNSEND, University of Illinois, Champaign, Ill. Three nominations for membership were received. The following papers were read:

(1) Mr. R. A. ROBERTS: "On the locus of the foci of conics having double contact with two fixed conics."

(2) Professor F. MORLEY: "On the common tangents to two similar epicycloids."

(3) Dr. G. W. HILL: "On the convergence of the series used in the subject of perturbations."

AT the Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY, which will be held on Friday afternoon, December 27, at three o'clock, the President, Dr. G. W. HILL, will deliver an address entitled: "*Some Remarks on the Progress of Celestial Mechanics since the Middle of the Century.*"

THE mathematical courses of the *Faculté des Sciences* in Paris during the first semester of the current academic year include the following: — Professor Darboux: Principles of geometry and theory of algebraic curves; — Professor Picard: Analogies between the theory of algebraic equations and that of linear differential equations; — Professor Appell: General laws of equilibrium and motion; — Professor Tisserand: Principal methods employed for the determination of orbits of comets, planets, and double stars; — Professor Poincaré: Study of elasticity; — Professor Boussinesq: Fluids; study of their mechanical properties, and the most important motions in which interior friction plays a secondary part. The following supplementary courses are also given: — Mr. Painlevé: Differential and integral calculus; — Mr. Koenigs: Kinematics; — Mr. Andoyer: Rotations of the heavenly bodies on themselves. Mathematical conferences are conducted by Messrs.

Painlevé, Puiseux, Raffy, Andoyer, and Koenigs. Professor Hermite will not lecture until the second semester.

AMONG the courses for the current winter semester at Göttingen are the following: — Professor Schering: Potential function; Magnetic observations; Mathematico-physical seminarium; — Professor Klein: Theory of numbers; Mathematical seminarium; — Professor Schur: Spherical astronomy, I.; Practical work with instruments in the observatory; Method of least squares; Astronomical problems in seminarium; — Professor Hilbert: Integral calculus; Theory of partial differential equations; Mathematical seminarium; — Professor Schoenflies: Theory of functions; Descriptive geometry; Mathematical proseminarium; — Dr. Burkhardt: Vector analysis; — Dr. Ambronn: Astrophysics; Fundamental theory of astronomy; — Dr. Pockels: Electromagnetic theory of light; Fundamental principles of modern meteorology; — Dr. Bohlmann: Homogeneous linear differential equations; Life insurance; — Dr. Sommerfeld: Projective geometry; Exercises in descriptive geometry.

AMONG the recent and forthcoming publications of A. Hermann are the following: J. BOLYAI: "La science absolue de l'éspace, indépendante de la vérité ou de la fausseté de l'axiome xi. d'Euclide," translated from the German by J. Houel. G. DARBOUX: "Sur une classe remarquable de courbes et de surfaces algébriques et sur la théorie des imaginaires," second edition, with notes and additions. G. DEMARTRES: "Cours d'analyse de la Faculté des Sciences de Lille; 3me partie; Équations différentielles et aux dérivées partielles." P. DUHEM: "Le potentiel thermodynamique et ses applications à la mécanique chimique et à l'étude des phénomènes électriques," second edition. E. GOURSAT: "Leçons sur l'intégration des équations aux dérivées partielles du second ordre." This work is to consist of two volumes, the first of which is now ready. P. PAINLEVÉ: "Leçons sur le frottement, professées à la Faculté des Sciences de Paris." B. RIEMANN: "Sur les hypothèses qui servent de fondement à la géométrie," translated from the German by J. Houel.

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MAYER (J.). Ueber $n$te Potenzreste und binomische Congruenzen dritten Grades. [Diss.] München, 1895. 8vo. 30 pp.

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